Annales Mathematicae et Informaticae
39 *(2012) pp. 83–108*

Proceedings of the Conference on Stochastic Models and their Applications
Faculty of Informatics, University of Debrecen, Debrecen, Hungary, August 22–24, 2011

Compensated compactness and relaxation at the microscopic level*

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

This is a survey of some recent results on hyperbolic scaling limits. In contract to diffusive models, the resulting Euler equations of hydrodynamics develop shocks in a finite time. That is why the derivation of the macroscopic equations from a microscopic model requires a synthesis of probabilistic and PDE methods. In the case of two-component stochastic models with a hyperbolic scaling law the method of compensated compactness seems to be the only tool that we can apply. Since the associated *Lax entropies* are not preserved by the microscopic dynamics, a logarithmic Sobolev inequality is needed to evaluate *entropy production.* Extending the arguments of Shearer (1994) and Serre–Shearer (1994) to stochastic systems, the nonlinear wave equation of isentropic elastodynamics is derived as the hyperbolic scaling limit of the anharmonic chain with Ginzburg–Landau type random perturbations. The model of interacting exclusion of charged particles results in the Leroux system in a similar way. In the presence of an additional creation-annihilation mechanism the missing logarithmic Sobolev inequality is replaced by an associated relaxation scheme. In this case the uniqueness of the limit is also known.

Keywords: Anharmonic chain, Ginzburg–Landau model, interacting exclusions, creation and annihilation, hyperbolic scaling, vanishing viscosity limit, logarithmic Sobolev inequalities, Lax entropy pairs, compensated compactness, relaxation schemes.

MSC: Primary 60K31, secondary 82C22.

*Partially supported by Hungarian Science Foundation Grants K-60708 and K-100473.

1. Historical notes and references

The idea that the Euler equations of hydrodynamics ought to be derived from statistical mechanics goes back to Morrey (1955). He proposed a *scaling limit* to pass to the hyperbolic system of classical conservation laws when the number of particles goes to infinity. The natural scaling of mechanical and related asymmetric systems is *hyperbolic:* the microscopic time is speeded up at the same rate at which the size of the system goes to infinity. The theory of *diffusive scaling limits* seems to be more or less complete, see Kipnis–Landim (1989) for a comprehensive survey.[1] Here we concentrate on the hyperbolic scaling limit of stochastic systems. Various models are introduced, and the main ideas of several proofs are also outlined in the next sections. You shall see that progress in this direction is rather slow, there are many relevant open problems.

Basic principles: In theoretical physics it is commonly accepted that the *equilibrium states* of the microscopic system are specified by the *Boltzmann–Gibbs formalism,* and the evolved measure can be well approximated by means of such Gibbs states with space and time dependent parameters. This *principle of local equilibrium* is used then to determine the macroscopic flux of the conserved quantities of the underlying microscopic dynamics; this is the first crucial problem in the theory of *hydrodynamic limits* (HDL). However, a rigorous verification of any version of this principle is problematic because the standard argument is based on a strong form of the *ergodic hypothesis,* which amounts to a description of translation invariant stationary states of the microscopic system as superpositions of the equilibrium Gibbs random fields. This is certainly one of the hardest open problems of mathematics, it is much more difficult than the question of *metric transitivity* of the underlying stationary process, but it is much weaker than the claim of the principle of local equilibrium. A second principal difficulty in the theory of hyperbolic scaling limits comes from the complexity of the resulting macroscopic equations (conservation laws). The breakdown of the existence of global classical solutions is quite general, and the surviving weak solutions are usually not unique. The formation of the associated *shock waves* results in extremely strong fluctuations at the microscopic level, too. Concerning terminology and basic facts on HDL we refer to the textbooks by Spohn (1991) and Kipnis–Landim (1999), while to Hörmander (1997), Bressan (2000) and Dafermos (2005) on PDE theory.

Deterministic models: Of course, there exist some mechanical systems that admit explicit computations. However, the exactly solvable models of *one-dimensional hard rods* and *coupled harmonic oscillators* are not ergodic in the traditional sense. Besides the classical ones these systems admit a continuum of conservation laws, consequently the scaling limit of such models does not result in a closed system of a finite set of equations for the classical conservation laws, see the papers by Dobrushin and coworkers (1980, 1983, 1985). The treatment of more realistic

[1] More recent information can be found on the web site http://stokhos.shinshu-u.ac.jp/10thSALSIS/ of the 10-th Symposium on Stochastic Analysis of Large Scale Interacting System, Kochi (Japan) 2011.

mechanical systems is out of question, Sinai (1988) is the only scientist who dared to attack this issue. He claimed that the identification of the macroscopic flux does not require the strong ergodic hypothesis, the problem is still open.

Attractive systems: To avoid the hopeless issue of strong ergodicity of mechanical systems, stochastic models are only considered in the rest of the related literature on hydrodynamic limits. Appropriately chosen random effects regularize the dynamics, thus there is a good chance to identify the conservation laws and the associated stationary states of the microscopic system. The first result in this direction is due to Rost (1981), he managed to derive certain rarefaction wave solutions to the Burgers equation as HDL of the totally asymmetric simple exclusion process. Following some preliminary studies by various authors, a few years later Rezakhanlou (1991) extended his *coupling technique* for a large class of *attractive models.* Several more recent results in this direction are treated or mentioned by Kipnis–Landim (1989) and Bahadoran (2004). Although the appearance of shocks is not excluded, effective coupling in attractive models implies the *Kruzkov entropy condition* in a natural way, consequently the empirical process converges to the uniquely specified *weak entropy solution* of the associated single conservation law. We are mainly interested in the hydrodynamic limit of microscopic systems with two conservation laws, these are certainly not attractive.

Entropy and HDL in a smooth regime: Random effects might regularize even the classical dynamics in such a way that we have a description of stationary measures: translation invariant equilibrium states of finite specific entropy with respect to a given stationary measure are all superpositions of the classical equilibrium (Gibbs) states. As a next step, a fairly general theory of *asymptotic preservation of local equilibrium* has been initiated by Yau (1991). This means that if the initial distribution is close to local equilibrium in the sense of specific relative entropy, then this property remains in force as long as the macroscopic solution is smooth enough. His method has been extended to Hamiltonian dynamics[2] with conservative noise for continuous particle systems by Olla–Varadhan–Yau (1993). The hyperbolic (Euler) scaling limit yields the full set of the *compressible Euler equations.* The basic ideas of this approach are to be discussed in the next section.

The problem of shocks: In the case of a hyperbolic scaling limit the microscopic system simply does not have enough time to organize itself, even the asymptotic preservation of local equilibrium is a problematic issue in a regime of *shock waves.* Therefore the separation of the *slowly varying* conserved quantities from the other, *rapidly oscillating* ones is less transparent than in a smooth or diffusive regime.

The existence theory of parabolic equations or systems is based on the associated *energy inequalities,* and it is a quite natural idea of PDE theory to construct a parabolic approximation to the hyperbolic system of conservation laws by adding elliptic (viscid) terms to the right hand side of the equations under consideration. Since the related energy inequalities degenerate in this *small viscosity limit,* the standard compactness argument has to be replaced by a radically new technique

[2]The kinetic energy of the model is not the classical one because energy transport can not be controlled in that case.

called *compensated compactness,* see Hörmander (1997) or Dafermos (2005) with several further references.

The microscopic models of hydrodynamics imitate this approach, thus the situation is quite similar. The probabilistic a priori bounds we have in a diffusive scaling limit[3] do not work any more in case of a hyperbolic scaling limit. Therefore we have to extend the theory of compensated compactness to our microscopic systems, see Fritz (2001, 2004, 2011), Fritz–Tóth (2004), Fritz–Nagy (2006) and Bahadoran–Fritz–Nagy (2011). In this way we obtain convergence along subsequences to *weak solutions,* and the uniqueness of the limit ought to be the consequence of some additional information. The familiar *Lax entropy inequality* is only sufficient for weak uniqueness to a single conservation law. Unfortunately, in the case of systems the much deeper Oleinik type conditions of Bressan (2000) are required, and these strictly local bounds are not attainable by our present probabilistic techniques.

2. The anharmonic chain

It is perhaps the simplest mechanical system that exhibits a correct physical behavior, it is considered as a *microscopic model of one-dimensional elasticity.* The Hamiltonian of coupled oscillators of unit mass on $\mathbb{Z}$ reads as

$$H(\omega) := \sum_{k\in\mathbb{Z}} H_k(\omega), \quad H_k(\omega) := p_k^2/2 + V(q_{k+1} - q_k),$$

where $\omega = \{(p_k, q_k) : k \in \mathbb{Z}\}$ denotes a configuration of the infinite system, $p_k, q_k \in \mathbb{R}$ are the momentum (velocity) and position of the oscillator at site $k \in \mathbb{Z}$. In terms of the *deformation* variables $r_k := q_{k+1} - q_k$, the equations of motion read as

$$\dot{p_k} = V'(r_k) - V'(r_{k-1}) \quad \text{and} \quad \dot{r_k} = p_{k+1} - p_k \quad \text{for } k \in \mathbb{Z}; \tag{2.1}$$

in this formulation the interaction potential V needs not be symmetric. The existence of unique solutions in a space of configurations $\omega := \{(p_k, r_k) : k \in \mathbb{Z}\}$ with a sub-exponential growth is quite standard if V' is Lipschitz continuous, i.e. if V'' is bounded. The related iterative procedure shows also that the solutions of the infinite system can be well approximated by the solutions of its finite subsystems when the size of the finite system goes to infinity, see e.g. Fritz (2011) with further references.

Although (2.1) is a direct lattice approximation to the *p-system* $\partial_t u = \partial_x V'(v)$, $\partial_t v = \partial_x u$, its convergence is rather problematic. In PDE theory (2.1) is not considered as a stable numerical scheme for solving the p-system, thus we can not believe in its convergence. The right way of its regularization is suggested by the small viscosity approach, it is certainly not difficult to define stable approximation schemes in this way. However, the theory of hydrodynamic limits goes beyond numerical analysis as discussed below.

[3] See Fritz (1986), and Guo–Papanicolau–Varadhan (1988) for a more perfect treatment.

2.1. The compressible Euler equations

(2.1) reads as a *lattice system of conservation laws* for the total momentum $P := \sum p_k$, and for the total deformation $R := \sum r_k$, respectively: $\partial_t P = \partial_t R = 0$ are formal identities. Since $\partial_t H_k(\omega) = p_{k+1}V'(r_k) - p_k V'(r_{k-1})$ is a difference of currents, the total energy H is also preserved by the dynamics, therefore we expect to have three hydrodynamic equations: one for momentum, one for the deformation, and one for energy. In view of the principle of local equilibrium, the macroscopic fluxes of these conservative quantities are to be calculated by means of the stationary states of the dynamics.

Stationary states and thermodynamics: These are characterized by $\int \mathcal{L}_0\varphi \, d\lambda = 0$ for smooth *local functions* φ of a finite number of variables, where

$$\mathcal{L}_0\varphi := \sum_{k\in\mathbb{Z}} \left((V'(r_k) - V'(r_{k-1}))\frac{\partial\varphi}{\partial p_k} + (p_{k+1} - p_k)\frac{\partial\varphi}{\partial r_k} \right) \tag{2.2}$$

denotes the associated *Liouville operator.* Assuming $\lim V(x)/|x| = +\infty$ as $|x| \to +\infty$, it is easy to check that we have a three-parameter family $\lambda_{\beta,\pi,\gamma}$ of translation invariant product measures: $\beta > 0$ is the *inverse temperature,* $\pi \in \mathbb{R}$ denotes the *mean velocity,* and $\gamma \in \mathbb{R}$ is a *chemical potential.* Under $\lambda_{\beta,\pi,\gamma}$ the marginal Lebesgue density of any couple $(p_k, r_k) \sim (y, x)$ reads as $\exp(\gamma x - \beta I(y, x|\pi) - F(\beta, \gamma))$, where $I(y, x|\pi) := (y - \pi)^2/2 + V(x)$; the normalization

$$F(\beta, \gamma) := \log \iint_{\mathbb{R}^2} \exp\left(\gamma r - \beta I(y, x|\pi)\right)\, dy\, dx \tag{2.3}$$

is sometimes referred to as the *free energy.* Indeed, approximating the infinite system by its finite subsystems, it follows immediately that these product measures are really equilibrium states of (2.1). It is easy to see that $\mathcal{L}_0$ is antisymmetric with respect to any $\lambda_{\beta,\pi,\gamma}$.

Let us remark that there is a one-to-one correspondence between the parameters (β, π, γ) and the corresponding expected values (h, u, v) of the conservative quantities H_k, p_k and r_k with respect to $\lambda_{\beta,\pi,\gamma}$. It is plain that $u := \int p_k \, d\lambda_{\beta,\pi,\gamma} = \pi$ is the mean velocity. By a direct computation we see also that the equilibrium mean of the *internal energy* $I_k := I(p_k, r_k|\pi)$ at one site is given by $\chi := \int I_k \, d\lambda_{\beta,\pi,\gamma} = -F'_\beta(\beta, \gamma)$, thus the equilibrium mean of the total energy $H_k = p_k^2/2 + V(r_k)$ is just $h := \chi + \pi^2/2$, while $v = F'_\gamma(\beta, \gamma) = \int r_k \, d\lambda_{\beta,\pi,\gamma}$ is the mean deformation. Integrating by parts we obtain $\int V'(r_k)\, d\lambda_{\beta,\pi,\gamma} = \gamma/\beta$ for the equilibrium expectation of V'. The parameters β and γ can be expressed in terms of the *thermodynamical entropy*

$$S(\chi, v) := \sup\left\{\gamma v - \beta\chi - F(\beta, \gamma) : \beta > 0, \gamma \in \mathbb{R}\right\} \tag{2.4}$$

as follows. Since S is the convex conjugate of F, we have $\gamma = S'_v(\chi, v)$ and $\beta = -S'_\chi(\chi, v)$ if $v = F'_\gamma(\beta, \gamma)$ and $\chi = -F'_\beta(\beta, \gamma)$.

The hyperbolic scaling limit: We are interested in the asymptotic behavior of the *empirical processes* $u_\varepsilon(t, x) := p_k(t/\varepsilon)$, $v_\varepsilon(t, x) := r_k(t/\varepsilon)$ and $h_\varepsilon(t, x) :=$

$H_k(\omega(t/\varepsilon))$ if $|k\varepsilon - x| < \varepsilon/2$, as $0 < \varepsilon \to 0$. Of course it is assumed that at time zero these processes converge, at least in a weak sense to the corresponding initial values of the hydrodynamic equations.

In view of the physical principle of local equilibrium, the macroscopic currents of the conservative quantities should be calculated by means of a product measure of type $\lambda_{\beta,\pi,\gamma}$ with parameters depending on time and space. In this framework $\gamma/\beta = \int V'(r_k)\,d\lambda_{\beta,\pi,\gamma}$ is the mean current of momentum, and $\pi\gamma/\beta = \int p_k V'(r_{k-1})\,d\lambda_{\beta,\pi,\gamma}$ is the mean current of energy, consequently a formal calculation results in the triplet of compressible Euler equations:

$$\partial_t u = \partial_x J(\chi, v), \quad \partial_t v = \partial_x u \quad \text{and} \quad \partial_t h = \partial_x(uJ(\chi, v)), \tag{2.5}$$

where $J(\chi, v) := \gamma/\beta = -S'_v(\chi, v)/S'_\chi(\chi, v)$ and $\chi = h - u^2/2$, see Chen–Dafermos (1995) and Fritz (2001). Therefore $\partial_t\chi = J(\chi, v)\partial_x u$ and $\partial_t S(\chi, v) = 0$ along classical solutions, but we have to keep in mind that this system develops shock waves in a finite time.

2.2. Stochastic perturbations

As we have emphasized before, we are not able to materialize the heuristic derivation of the compressible Euler equations, the dynamics of the anharmonic chain should be regularized by a well chosen noise. There are several plausible tricks, we are going to consider Markov processes generated by an operator $\mathcal{L} = \mathcal{L}_0 + \sigma\,\mathcal{G}$, where $\mathcal{L}_0$ is the Liouville operator, while the Markov generator $\mathcal{G}$ is symmetric in equilibrium. Here $\sigma > 0$ may depend on the scaling parameter $\varepsilon > 0$, and $\varepsilon\sigma(\varepsilon)$ is interpreted as the coefficient of *macroscopic viscosity.* We are assuming that $\varepsilon\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, then the effect of the symmetric component $\sigma\mathcal{G}$ diminishes in the limit. Our philosophy consists in adapting the *vanishing viscosity approach* of PDE theory to the microscopic theory of hydrodynamics. In a regime of shocks an additional technical condition: $\varepsilon\sigma^2(\varepsilon) \to +\infty$ is also needed.

Random exchange of velocities: As far as I understand, this is the weakest but still effective conservative noise. At the bonds of $\mathbb{Z}$ we have independently running clocks with exponential waiting times of parameter 1, and we exchange the velocities at the ends of the bond when the clock rings. The generator $\mathcal{G} = \mathcal{G}_{ep}$ of this exchange mechanism is acting on local functions as

$$\mathcal{G}_{ep}\varphi(\omega) = \sum_{k\in\mathbb{Z}} \left(\varphi(\omega^{k,k+1}) - \varphi(\omega)\right), \tag{2.6}$$

where $\omega^{k,k+1}$ denotes the configuration obtained from $\omega = \{(p_j, r_j)\}$ by exchanging p_k and p_{k+1}, the rest of ω remains unchanged. It is plain that $P = \sum p_k$, $R = \sum r_k$ and the total energy H are formally preserved by $\mathcal{G}_{ep}$, and the product measures $\lambda_{\beta,\pi,\gamma}$ are all stationary states of the Markov process generated by $\mathcal{L} := \mathcal{L}_0 + \sigma\,\mathcal{G}_{ep}$ if $\sigma > 0$.

This model was introduced by Fritz–Funaki–Lebowitz (1994), where the strong ergodic hypothesis is proven for lattice models with two conservation laws. The proof applies also in our case without any essential modification, see below. The *relative entropy* $\mathsf{S}[\mu|\lambda]$ of two probability measures on the same space is defined by $\mathsf{S} := \int \log f \, d\mu$, provided that $f = d\mu/d\lambda$ and the integral does exist; $\mathsf{S}[\mu|\lambda] = +\infty$ otherwise.[4] Let μ_n denote the joint distribution of the variables $\{(p_k, r_k) : |k| \le n\}$ with respect to μ, as a reference measure we choose $\lambda := \lambda_{1,0,0}$, and $f_n := d\mu_n/d\lambda$.

Theorem 2.1. *Suppose that μ is a translation invariant stationary measure of the process generated by $\mathcal{L} = \mathcal{L}_0 + \sigma \mathcal{G}_{ep}$. If the specific entropy of μ is finite, i.e. $\mathsf{S}[\mu_n|\lambda] = O(n)$, then μ is contained in the weak closure of the convex hull of our set $\{\lambda_{\beta,\pi,\gamma}\}$ of stationary product measures.*

On the ideas of the proof: The basic steps can be outlined as follows, for technical details see Theorems 2.4 and 3.1 of our paper cited above, or an improved version of the notes by Bernardin–Olla (2010). Since $\mathsf{S}[\mu_n|\lambda]$ is constant in a stationary regime, $\int \mathcal{L} \log f_n \, d\mu = 0$. The contribution of $\mathcal{L}_0$ consists of two boundary terms only because $\mathcal{L}_0$ is antisymmetric, while $-D_n[\mu|\lambda]$ is the essential part of the contribution of the symmetric $\mathcal{G}_{ep}$, where $D_n := -\int f_n \mathcal{G}_{ep} \log f_n \, d\lambda$. Due to the translation invariance of μ we see immediately that $(1/n) D_n[\mu|\lambda] \to 0$ as $n \to +\infty$. Moreover, $D_n \ge 0$ is a convex functional of μ, thus $D_{n+m} \ge D_n + D_m$, whence even $D_n[\mu|\lambda] = 0$ follows for all $n \in \mathbb{N}$. Therefore μ is symmetric with respect to any exchange of velocities, i.e. $\int \mathcal{G}_{ep} \varphi \, d\mu = 0$ is an identity, consequently the stationary Liouville equation $\int \mathcal{L}_0 \varphi \, d\mu = 0$ also holds true.

Let $\phi(p)$ and $\psi(r)$ denote local functions depending only on the velocity and the deformation variables $p := \{p_j\}$, $r := \{r_j\}$, respectively. If φ_k and ψ_k are their translates by $k \in \mathbb{Z}$, then

$$\int \phi_k(p)\psi_k(r) \, d\mu = \int \phi_k(p)\psi_0(r) \, d\mu = \frac{1}{l} \sum_{j=0}^{l-1} \int \phi_{k+j}(p)\psi_0(r) \, d\mu$$

are identities, and the law of large numbers applies to the right hand side. For instance we see that given r, the conditional distribution of p is exchangeable, and it does not depend on the individual deformation variables r_j, thus the conditional expectation of any p_j is an invariant and tail measurable function $u \sim \pi$. Similarly, the conditional variance Q of velocities defines our first parameter, the inverse temperature β by $\beta := 1/Q$, it is an invariant function, too. Moreover, the entropy condition implies $\beta > 0$ almost surely.

On the other hand, for $\varphi = \psi(r)(p_k - u)$ the stationary Liouville equation yields

$$\int \psi(r)(V'(r_k) - V'(r_{k-1}) \, d\mu = \sum_{j \in \mathbb{Z}} \int \frac{\partial \psi(r)}{\partial r_j} \, (p_k - u)(p_{j+1} - p_j)) \, d\mu.$$

In view of the De Finetti–Hewitt–Savage theorem, the velocities are conditionally independent when r is given, consequently

$$\int \psi(r) \left(V'(r_k) - V'(r_{k-1})\right) \, d\mu = \int \frac{1}{\beta} \left(\frac{\partial \psi}{\partial r_k} - \frac{\partial \psi}{\partial r_{k-1}} \right) \, d\mu.$$

[4]The *entropy inequality* $\int \varphi \, d\mu \le \mathsf{S}(\mu|\lambda) + \log \int e^{\varphi} \, d\lambda$ is used in several probabilistic computations; $\varphi = \log f$ is the condition equality.

Now an obvious summation trick lets the law of large numbers work, whence

$$\int \psi(r)(V'(r_k) - \gamma)\, d\mu = \int \frac{1}{\beta} \frac{\partial \psi(r)}{\partial r_k}\, d\mu,$$

where the parameter γ is again invariant and tail measurable because it is the limit of the arithmetic averages of the $V'(r_j)$ variables. The stationary Liouville equation has been separated (localized) in this way, therefore the distribution of the deformation variables can be identified. Indeed, as β does not depend on r_k, the desired statement reduces to the differential characterization of the Lebesque measure by integrating by parts. In the case of velocities a similar argument results in

$$\int \phi(p)(p_k - \pi)\, d\mu = \int \frac{1}{\beta} \frac{\partial \phi(p)}{\partial p_k}\, d\mu,$$

consequently if the tail field is given, then the conditional distribution of $\omega = \{(p_k, r_k)\}$ under μ is just $\lambda_{\beta,\pi,\gamma}$.

It is interesting to note that Theorem 2.1 is not true for finite systems because the cited theorem on exchangeable variables applies to infinite sequences only.

Physical viscosity with thermal noise: Another popular model is obtained by adding a *Ginzburg-Landau type conservative noise* to the equations of velocities:

$$\begin{aligned} dp_k &= (V'(r_k) - V'(r_{k-1}))\, dt + \sigma\, (p_{k+1} + p_{k-1} - 2p_k)\, dt \\ &\quad + \sqrt{2\sigma}\, (dw_k - dw_{k-1}), \quad dr_k = (p_{k+1} - p_k)\, dt, \quad k \in \mathbb{Z}, \end{aligned} \tag{2.7}$$

where $\sigma > 0$ is a given constant, and $\{w_k : k \in \mathbb{Z}\}$ is a family of independent Wiener processes. Due to $V'' \in L^\infty$, the existence of unique strong solutions to this infinite system of stochastic differential equations is not a difficult issue, see e.g. Fritz (2001) with further references. The generator of the Markov process defined in this way can again be written as $\mathcal{L} := \mathcal{L}_0 + \sigma \mathcal{G}_p$, where $\mathcal{G}_p$ is now an elliptic operator. Total energy is not preserved any more, and a thermal equilibrium of unit temperature is maintained by the noise. It is easy to check that the product measures $\lambda_{\pi,\gamma} := \lambda_{1,\pi,\gamma}$ are all stationary, thus (2.5) reduces to the *p-system* (nonlinear sound equation) of elastodynamics:

$$\partial_t u = \partial_x S'(v) \quad \text{and} \quad \partial_t v = \partial_x u, \quad \text{that is} \quad \partial_t^2 v = \partial_x^2 S'(v) \tag{2.8}$$

because $\int V'(r_k)\, d\lambda_{\pi,\gamma} = \gamma = S'(v)$ if $\int r_k\, d\lambda_{\pi,\gamma} = v = F'(\gamma)$, where

$$S(v) := \sup_\gamma \{\gamma v - F(\gamma)\}; \quad F(\gamma) := \log \int_{-\infty}^{\infty} \exp(\gamma x - V(x))\, dx.$$

Let us remark that both F and S are infinitely differentiable, and $S''(v) = 1/F''(\gamma)$ is strictly positive and bounded.

The verification of the strong ergodic hypothesis is similar, but considerably simpler than in the previous case:

Theorem 2.2. *Translation invariant stationary measures of finite specific entropy are superpositions of our product measures $\lambda_{\pi,\gamma}$.*

For a complete proof see Theorem 13.1 in the notes by Fritz (2001). HDL of this model follows easily by the relative entropy argument of Yau. At a level $\varepsilon > 0$ of scaling $\mu_{t,\varepsilon,n}$ denotes the true distribution of the variables $\{(p_k(t), r_k(t)) : |k| \le n\}$, and $\lambda_{t,\varepsilon} \sim \lambda_{\pi,\gamma}$ is a product measure with parameters $\pi = \pi_k(t,\varepsilon)$ and $\gamma = \gamma_k(t,\varepsilon)$ depending on space and time. We say that asymptotic local equilibrium holds true on the interval $[0,T]$ if we have a family $\{\lambda_{t,\varepsilon} : t \le T/\varepsilon, \varepsilon \in (0,1]\}$ such that for all $\tau \le T$

$$\lim_{\varepsilon\to 0} \sup_{n\ge 1/\varepsilon} \frac{\mathsf{S}[\mu_{\tau/\varepsilon,\varepsilon,n}|\lambda_{\tau/\varepsilon,\varepsilon}]}{2n+1} = 0. \tag{2.9}$$

Postulate this for $\tau = 0$, and suppose also that the prescribed initial values give rise to a continuously differentiable solution (u,v) to (2.8) on $[0,T], T > 0$. Then the approximate local equilibrium (2.9) remains in force for $\tau \le T$, at least if the parameters π_k and γ_k of $\lambda_{t,\varepsilon}$ are chosen in a clever way, namely as they are predicted by the hydrodynamic equations (2.8). For example, we can put $\pi_k(t,\varepsilon) := u(\tau/\varepsilon, k/\varepsilon)$ and $\gamma_k(t,\varepsilon) := S'(v(\tau/\varepsilon, k/\varepsilon))$ if $t = \tau/\varepsilon$, but solutions to a discretized version of (2.8) can also be used. Therefore the empirical processes u_ε and v_ε converge in a weak sense to that smooth solution of (2.8). Indeed, the entropy inequality implies $-\log\lambda[A]\,\mu[A] \le \mathsf{S}[\mu|\lambda] + \log 2$ for any event A, and in an exact local equilibrium $\lambda_{t,\varepsilon}$ the weak law of large numbers holds true with an exponential rate of convergence. Consequently (2.9) implies

Theorem 2.3. *Under the conditions listed above we have*

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \varphi(x) u_\varepsilon(\tau,x)\,dx = \int_{-\infty}^{\infty} \varphi(x) u(\tau,x)\,dx$$

and

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \psi(x) v_\varepsilon(\tau,x)\,dx = \int_{-\infty}^{\infty} \psi(x) v(\tau,x)\,dx$$

in probability for all continuous φ, ψ with compact support if $\tau \le T$, where (u,v) is the preferred smooth solution to (2.8).

The main ideas concerning the derivation of (2.9) are discussed in the next subsection, for a complete proof see that of Theorem 14.1 in Fritz (2001). In contrast to the result of Olla–Varadhan–Yau (1993) and other related papers, see also Theorem 2.4 below, the statement is not restricted to the periodic setting; the scaling limit here is considered on the infinite line. Such an extension of the original argument is based on the observation that the boundary terms of $\partial_t\mathsf{S}[\mu_{t,\varepsilon,n}|\lambda_{t,\varepsilon}]$ can be controlled by the associated *Dirichlet form* consisting of the volume terms of $\partial_t\mathsf{S}$. The first proof in this direction is due to Fritz (1990), see also Fritz–Nagy (2006), Bahadoran–Fritz–Nagy (2011) and Fritz (2011).

2.3. Derivation of the Euler equations in a smooth regime

Here we are going to outline Yau's method for the anharmonic chain with random exchange of velocities. The argument is similar but much more transparent than that of Olla–Varadhan–Yau (1993). The derivation of (2.8) is easier, its main steps are also included in the next coming calculations. Since the noise is not strong enough to control the flux of the relative entropy, we have to formulate the problem in a periodic setting: $p_k(0) = p_{k+n}(0)$ and $r_k(0) = r_{k+n}(0)$ for all k with some $n \in \mathbb{N}$. The evolved configuration remains periodic for all times, which means that the system can be considered on the discrete circle of length $n \to +\infty$. The coefficient $\sigma > 0$ can be kept fixed during the procedure of scaling because the only role of the exchange mechanism is to ensure the strong ergodic hypothesis. At a level $\varepsilon = 1/n$ of scaling let $\mu_{t,n}$ denote the evolved measure, and consider the *local equilibrium distributions* $\lambda_{t,n}$ of type $\lambda_{\beta,\pi,\gamma}$ with parameters depending on space and time: $\beta = \beta_k(t,n)$, $\pi = \pi_k(t,n)$ and $\gamma = \gamma_k(t,n)$.

Theorem 2.4. *Suppose that $(1/n)\mathsf{S}[\mu_{0,n}|\lambda_{0,n}] \to 0$ as $n \to +\infty$, and the related initial values determine a smooth solution (u,v,h) to (2.5) on the interval $[0,T]$ of time such that $\beta = -S'_\chi(\chi,v)$ remains strictly positive. Then*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \psi(x) z_n(t,x)\,dx = \int_{-\infty}^{\infty} \psi(x) z(t,x)\,dx$$

in probability for all continuous ψ with compact support if $t \le T$, where (z_n, z) is any of the couples (u_n,u), (v_n,v), (h_n,h), and $u_n(t,x) := p_k(tn)$, $v_n(t,x) := r_k(tn)$, $h_n(t,x) := H_k(tn)$ if $|k - xn| < 1/2$.

In view of the argument we have sketched before Theorem 2.3, we have to show that if the parameters of $\lambda_{t,n}$ are defined by means of the smooth solution, then $(1/n)\mathsf{S}[\mu_{\tau n,n}|\lambda_{\tau n,n}] \to 0$ as $n \to +\infty$ for all $\tau \le T$, consequently the empirical processes converge in a weak sense to that solution of (2.5).

Calculation of entropy: Let $f_{t,n} := d\mu_{t,n}/d\lambda_{t,n}$ and consider the time evolution of $\mathsf{S}[\mu_{t,n}|\lambda_{t,n}] = \int \log f_{t,n}\,d\mu_{t,n}$. In the next coming calculations we are assuming that the evolved density $f_{t,n}(\omega) > 0$ is a continuously differentiable function. This hypothesis can be relaxed by means of a standard regularization procedure, see e.g. Fritz–Funaki–Lebowitz (1994). The required regularity of the parameters is a consequence of their construction via discretizing the macroscopic system (2.5). By a formal computation

$$\partial_t \mathsf{S}[\mu_{t,n}|\lambda_{t,n}] = \int (\partial_t + \mathcal{L}_0 + \sigma\mathcal{G}_{ep}) \log f_{t,n}\,d\mu_{t,n} \le \int (\partial_t + \mathcal{L}_0) f_{t,n}\,d\lambda_{t,n}$$

because $\int f_{t,n}\,d\lambda_{t,n} \equiv 1$, $\mathcal{L}_0 \log f_{t,n} = (1/f_{t,n})\mathcal{L}_0 f_{t,n}$, and the contribution of $\mathcal{G}_{ep}$ is certainly not positive. Moreover, as $\mathcal{L}_0$ is antisymmetric with respect to the Lebesgue measure, we have

$$\int (\partial_t f_{t,n} + f_{t,n}\partial_t \log g_{t,n})\,d\lambda_{t,n} = \int (\mathcal{L}_0 f_{t,n} + f_{t,n}\mathcal{L}_0 \log g_{t,n})\,d\lambda_{t,n} = 0,$$

where $g_{t,n}$ denotes the Lebesgue density of $\lambda_{t,n}$, consequently

$$\mathsf{S}[\mu_{t,n}|\lambda_{t,n}] \leq \mathsf{S}[\mu_{0,n}|\lambda_{0,n}] - \int_0^t \int (\partial_s + \mathcal{L}_0) \log g_{s,n} \, d\mu_{s,n} \, ds. \tag{2.10}$$

On the other hand, as

$$\log g_{s,n} = \sum_{k=0}^{n-1} (\gamma_k r_k - \beta_k I_k - F(\beta_k, \gamma_k)),$$

where $I_k := I(p_k, r_k|\pi_k) = (p_k - \pi_k)^2/2 + V(r_k)$, by a direct calculation we obtain that

$$\partial_s \log g_{s,n} = \sum_{k=0}^{n-1} \Big(\dot{\gamma}_k (r_k - v_k) + \beta_k \dot{\pi}_k (p_k - \pi_k) - \dot{\beta}_k (I_k - \chi_k) \Big),$$

where "dot" indicates differentiation with respect to time.

There is a fundamental relation between the parameters β, π, γ of $\lambda_{n,t}$, namely

$$\sum_{k=0}^{n-1} ((\gamma_{k-1} - \gamma_k)\pi_k + (\beta_k \pi_k - \beta_{k+1}\pi_{k+1}) J_k + (\beta_{k+1} - \beta_k)\pi_{k+1} J_k) = 0.$$

As it is explained by Tóth–Valkó (2003), this identity is due to the conservation of the *thermodynamic entropy* in a smooth regime, which is a basic feature of all models with a proper physical motivation. On the other hand, it is a necessary requirement when we evaluate the rate of production of S in order to conclude (2.9). Indeed, we get

$$\begin{aligned}\mathcal{L}_0 \log g_{s,n} = &\sum_{k=0}^{n-1} (\gamma_{k-1} - \gamma_k)(p_k - \pi_k) \\ &+ \sum_{k=0}^{n-1} (\beta_k \pi_k - \beta_{k+1}\pi_{k+1})(V'(r_k) - J_k) \\ &+ \sum_{k=0}^{n-1} (\beta_{k+1} - \beta_k)(p_{k+1} V'(r_k) - \pi_{k+1} J_k),\end{aligned} \tag{2.11}$$

where $v_k := \int r_k \, d\lambda_{t,n}$, $\pi_k := u_k = \int p_k \, d\lambda_{t,n}$ and $\chi_k := \int I_k \, d\lambda_{t,n}$, finally $J_k = J(\chi_k, v_k) = \gamma_k/\beta_k := \int V'(r_k) \, d\lambda_{t,n}$. Notice that the local equilibrium mean of any of the last factors on the right hand sides of (2.11) above does vanish: for instance $\int (V'(r_k) - J_k) \, d\lambda_{t,n} = 0$.

The crucial step: The microscopic time t is as big as $t = n\tau$, thus there is a danger of explosion on the right hand side of (2.10) as $n \to +\infty$. However, due to the smoothness of the macroscopic solution, the nonlinear functions appearing in the sums above can be substituted by their block averages, and the celebrated

One-block Lemma, which is the main consequence of strong ergodicity, allows us to approximate the block averages by their *canonical equilibrium expectations*, see Lemma 3.1 in Guo–Papanicolau–Varadhan (1988) or Theorem 3.5 of Fritz (2001).

The wave equation: The case of (2.8) is quite simple because $\beta_k \equiv 1$ then, thus $V_k' = V'(r_k)$ is the only nonlinear function we are facing with. Block averages $\bar\eta_{l,k} := (1/l)(\eta_k + \eta_{k-1} + \cdots + \eta_{k-l+1})$ of size $l \in \mathbb{N}$ are also periodic functions of $k \in \mathbb{Z}$ with period n. Since $\int V_k'\,d\lambda_{1,\pi,\gamma} = S'(v_k) = J_k$ if v_k is the local equilibrium mean of r_k, $\bar V_{l,k}' \approx S'(\bar r_{l,k})$ is the desired substitution, which is valid as $l \to +\infty$ after $n \to \infty$. Presupposing $|\pi_{k+1} - \pi_k| = O(1/n)$ and $|v_{k+1} - v_k| = O(1/n)$ we write

$$\sum_{k=0}^{n-1}(\pi_k - \pi_{k+1})(V'(r_k) - S'(v_k)) \approx \sum_{k=0}^{n-1}(\pi_k - \pi_{k+1})(\bar V_{l,k}' - S'(v_k))$$
$$\approx \sum_{k=0}^{n-1}(\pi_k - \pi_{k+1})(S'(\bar r_{l,k}) - S'(v_k)) \approx \sum_{k=0}^{n-1}(\pi_k - \pi_{k+1})S''(v_k)(r_k - v_k).$$

The remainders including the squared differences coming from the expansion of $S'(\bar r_{l,k}) - S'(v_k)$ are estimated by means of the basic entropy inequality and the related large deviation bound; let us omit these technicalities. Comparing the leading terms we see that

$$\dot\gamma = S''(v_k)(\pi_{k+1} - \pi_k) \quad \text{and} \quad \dot\pi_k = \gamma_k - \gamma_{k-1}$$

is the right choice of the parameters because then there is a radical cancelation on the right hand side of (2.10). Since $\gamma_k = S'(v_k)$, this system is just a lattice approximation to (2.8), thus our conditions on the regularity of the parameters are also justified. Summarizing the calculations above, we get a bound

$$\mathsf{S}[\mu_{\tau,n}|\lambda_{t,n}] \le \mathsf{S}[\mu_{0,n}|\lambda_{0,n}] + \frac{K}{n}\int_0^t \mathsf{S}[\mu_{s,n}|\lambda_{s,n}]\,ds + R_n(T,l) \tag{2.12}$$

such that $R_n(T,l) \to 0$ as $n \to +\infty$ and then $l \to +\infty$, whence $\mathsf{S}[\mu_{\tau n,n}|\lambda_{\tau n,n}] = o(n)$ follows by the Grönwall inequality if $\tau \le T$.

The general case: It is a bit more complicated then the case of the p-system, the required substitutions read as

$$V'(r_k) \approx J(\bar I_{l,k}, \bar r_{l,k}) \approx J_k + J_\chi'(\chi_k, v_k)(\bar I_{l,k} - \chi_k) + J_v'(\chi_k, v_k)(\bar r_{l,k} - v_k),$$

and

$$\begin{aligned} p_{k+1}V'(r_k) \approx \bar p_{l,k+1}J(\bar I_{l,k}, \bar r_{l,k}) &\approx \pi_{k+1}J(\chi_k, v_k) \\ &+ J(\chi_k, v_k)(\bar p_{l,k+1} - \pi_{k+1}) \\ &+ \pi_{k+1}J_\chi'(\chi_k, v_k)(\bar I_{l,k} - \chi_k) + \pi_{k+1}J_v'(\chi_k, v_k)(\bar r_{l,k} - v_k). \end{aligned}$$

These steps are justified by the strong ergodicity of the dynamics (One-block Lemma), provided that $V'(r_k)$ and $\pi_{k+1}V'(r_k)$ can be replaced by their block averages. This second condition turns out to be a consequence of the smoothness of the macroscopic solution, see the construction below. The second order quadratic terms of the expansions above are estimated by means of the entropy inequality, we only need standard large deviation bounds.

To minimize $\mathsf{S}[\mu_{t,n}|\lambda_{t,n}]$, the parameters of $\lambda_{t,n}$ should be defined by means of a discretized version of the Euler equations. In fact we set

$$\pi_k = u_k, \quad \gamma_k = S'_v(\chi_k, v_k), \quad \beta_k = -S'_\chi(\chi_k, v_k),$$

where

$$\dot{v}_k = u_{k+1} - u_k, \quad \dot{u}_k = J(\chi_{k+1}, v_{k+1}) - J(\chi_k, v_k)$$

and $\dot{\chi}_k = J(\chi_k, v_k)(u_{k+1} - u_k)$, whence

$$\begin{aligned}
\beta_k \dot{\pi}_k &= (\gamma_k - \gamma_{k-1}) + (\beta_{k-1} - \beta_k)\, J_{k-1}, \\
\dot{\gamma}_k &= (\beta_{k+1}\pi_{k+1} - \beta_k \pi_k) J'_v(\chi_k, v_k) + (\beta_k - \beta_{k+1}) J'_v(\chi_k, v_k), \\
\dot{\beta}_k &= (\beta_k \pi_k - \beta_{k+1}\pi_{k+1}) J'_\chi(\chi_k, v_k) + (\beta_{k+1} - \beta_k) J'_\chi(\chi_k, v_k)
\end{aligned}$$

follow by a direct computation.

As a consequence of these calculations, we see the expected cancelation of the sum of all leading terms on the right hand side of (2.10), while the remainders can be estimated by means of the entropy inequality. The summary of these computations results in (2.12), thus the proof can be terminated as it was outlined in the previous two paragraphs.

3. Compensated compactness via artificial viscosity

As we have already explained, randomness in the above modifications of the anharmonic chain implies convergence to a classical solution of the macroscopic system (2.5) or (2.8) by the strong ergodic hypothesis, but in a regime of shocks much more information is needed to pass to the hydrodynamic limit. Effective coupling techniques that we have for attractive models are not available in the case of two-component systems, compensated compactness seems to be the only tool we can use. The microscopic dynamics can not admit non-classical conservation laws because it should be ergodic in the strong sense, therefore a nontrivial Lax entropy is not conserved by the microscopic dynamics. In general, the flux of a Lax entropy exhibits a *non-gradient behavior*, but the standard spectral gap estimates of Varadhan (1994) are not sufficient for its control in this case, a *logarithmic Sobolev inequality* (LSI) is needed. The effective LSI is due to the *strong artificial viscosity* of our next model, we will consider a Ginzburg–Landau type stochastic system:

$$\begin{aligned}
dp_k &= (V'(r_k) - V'(r_{k-1}))\, dt + \sigma(\varepsilon)\,(p_{k+1} + p_{k-1} - 2p_k)\, dt \\
&\quad + \sqrt{2\sigma(\varepsilon)}\,(dw_k - dw_{k-1})
\end{aligned}$$

and

$$dr_k = (p_{k+1} - p_k)\,dt + \sigma(\varepsilon)\,(V'(r_{k+1}) + V'(r_{k-1}) - 2V'(r_k))\,dt \\ + \sqrt{2\sigma(\varepsilon)}\,(d\tilde{w}_{k+1} - d\tilde{w}_k),$$

where $\{w_k : k \in \mathbb{Z}\}$ and $\{\tilde{w}_k : k \in \mathbb{Z}\}$ are independent families of independent Wiener processes. Of course, the macroscopic viscosity $\varepsilon\sigma(\varepsilon)$ vanishes as $\varepsilon \to 0$, but we also need $\varepsilon\sigma^2(\varepsilon) \to +\infty$ to suppress extreme fluctuations of Lax entropies. To have a standard existence and uniqueness theory for this infinite system of stochastic differential equations, we are assuming that V'' is bounded. The generator of the *Feller process* defined in this way reads as $\mathcal{L} = \mathcal{L}_0 + \sigma\mathcal{G}_p + \sigma\mathcal{G}_r$, where $\mathcal{G}_r$ is also elliptic. Additional conditions on the interaction potential V are listed below.

3.1. Conditions and main result

Just as in the case of (2.7), the same $\{\lambda_{\pi,\gamma} : \pi, \gamma \in \mathbb{R}\}$ is the family of stationary product measures, and the converse statement, i.e. the strong ergodic hypothesis can be proven in the same way. Therefore again (2.8) is expected to govern the macroscopic behavior of the system under hyperbolic scaling. The first crucial problem is the evaluation of $\mathcal{L}_0 h$ when h is a Lax entropy, we have to show that its dominant part is a difference of currents. These probabilistic calculations are based on a logarithmic Sobolev inequality. In view of the *Bakry–Emery criterion,* see Deuschel–Stroock (1989), we have to assume that V *is strictly convex*, i.e. V'' is bounded away from zero. On the other hand, the existence of weak solutions to (2.8) requires the condition of *genuine nonlinearity:* the third derivative S''' can not have more than one root, see DiPerna (1985), Shearer (1994) and Serre–Shearer (1994). In terms of V this is a consequence of one of the following assumptions.

(i) V' is strictly convex or concave on $\mathbb{R}$.

(ii) V is symmetric and $V'(r)$ is strictly convex or concave for $r > 0$.

The very same properties of the flux S' follow immediately by the theory of *total positivity.* Of course, small perturbations of such potentials also imply the required genuine nonlinearity of the macroscopic flux, $V(r) := r^2/2 - a\log\cosh(br)$ is an explicitly solvable example if $a > 0$ is small enough.

A technical condition: *asymptotic normality* requires the existence of positive constants α, V''_+, V''_- and R such that $|V''(r) - V''_+| \le e^{-\alpha r}$ if $r \ge R$, while $|V''(r) - V''_-| \le e^{\alpha r}$ if $r \le -R$.

Since we are not able to prove the uniqueness of the hydrodynamic limit, our only hypothesis on the initial distribution is an entropy bound: $\mathsf{S}[\mu_{0,\varepsilon,n}|\lambda_{0,0}] = O(n)$.

Let P_ε denote the distribution of the empirical process $(u_\varepsilon, v_\varepsilon)$, then the simplest version of our main result reads as:

Theorem 3.1. P_ε *is a tight family with respect to the weak local topology of the L^2 space of trajectories, and its limit distributions are all concentrated on a set of weak solutions to* (2.8)*.*

The notion of weak convergence changes from step to step of the argument. We start with the *Young measure* of the block-averaged process, and at the end we get tightness in the strong local $L^p(\mathbb{R}^2_+)$ topology for $p < 2$; $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}$. This strong form of our result is proven for a mollified version $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ of the empirical process, it is defined a bit later, after (3.2). Compensated compactness is the most relevant keyword of the proofs.

3.2. On the ideas of the proof

We follow the argumentation of the vanishing viscosity approach. In a concise form (2.8) can be written as $\partial_t z + \partial_x \Phi(z) = 0$, where $z := (u, v)$, $\Phi(z) := -(S'(v), u)$, and its viscid approximation reads as $\partial_t z_\delta + \partial_x \Phi(z_\delta) = \delta\, \partial_x^2 z_\delta$. This parabolic system admits classical solutions if $\delta > 0$, and the original hyperbolic equation can be solved by sending $\delta \to 0$. The argument is not trivial at all, see e.g. Dafermos (2005). Our task is to extend this technology to microscopic systems.

Energy inequality: Observe first that the space integral of $W(z) := u^2/2 + S(v)$ is constant along classical solutions to the wave equation (2.8), moreover its viscid approximation satisfiess

$$\begin{aligned}\partial_t W(z_\delta) &= \partial_x \left(u_\delta S'(v_\delta)\right) + \delta\, \partial_x \left(u_\delta \partial_x u_\delta + S'(v_\delta) \partial_x v_\delta\right) \\ &\quad - \delta \left((\partial_x u_\delta)^2 + S''(v_\delta)(\partial_x v_\delta)^2\right).\end{aligned}$$

Since S is strictly convex, we have got a standard energy inequality: an L^2 bound for $\delta^{1/2}\, \partial_x z_\delta$. In a regime of shocks, however, this bound does not vanish as $\delta \to 0$, consequently a strong compactness argument is not available.

Young family: Nevertheless, a very weak form of compactness holds true at the level of the Young measure. The approximate solution z_δ can be represented by a measure Θ_δ on $\mathbb{R}^2_+ \times \mathbb{R}^2$ such that $d\Theta_\delta := dt\, dx\, \theta^\delta_{t,x}(dz)$, where $\theta^\delta_{t,x}$ is the *Dirac mass* sitting at the actual value $z_\delta(t, x)$ of z_δ. Since z_δ is locally bounded in $L^2(\mathbb{R}^2_+)$, we can select weakly convergent sequences from Θ_δ as $\delta \to 0$. Of course, the *Young family* $\{\theta_{t,x} : (t, x) \in \mathbb{R}^2_+\}$ of a limiting measure Θ of Θ_δ needs not be Dirac, thus we only have convergence to *measure valued solutions:* $\partial_t \theta_{t,x}(z) + \partial_x(\theta_{t,x}(\Phi(z))) = 0$ in the sense of distributions, where the abbreviation $\theta_{t,x}(\varphi(z)) := \int \varphi(z)\, \theta_{t,x}(dz)$ is used; we write $\theta_{t,x}(z)$ if $\varphi(z) \equiv z$. F The identification of measure valued solutions as weak solutions is the subject of the theory of compensated compactness, in fact the Dirac property of the limiting Young measure should be verified.

Compensated factorization: It is crucial that (2.8) admits a rich family of *Lax entropy pairs* (h, J), these are characterized by the conservation law: $\partial_t h(z) + \partial_x J(z) = 0$ along classical solutions. Let us now turn to the viscid approximation. We see that *entropy production*

$$\begin{aligned}X_\delta := \partial_t h(z_\delta) + \partial_x J(z_\delta) &= \delta\, \partial_x (h'_u \partial_x u_\delta + h'_v \partial_x v_\delta) \\ &\quad - \delta\, \left(h''_{uu}(\partial_x u_\delta)^2 + 2h''_{uv} \partial_x u_\delta\, \partial_x v_\delta + h''_{vv}(\partial_x v_\delta)^2\right)\end{aligned}$$

decomposes as $X_\delta = Y_\delta + Z_\delta$, where Y_δ vanishes in H^{-1}, while Z_δ is bounded in the space of measures. As a first consequence we get the *Lax entropy inequality*:

$X_\delta \leq 0$ as a distribution if h is convex, but the famous *Div-Curl Lemma* is more relevant at this point. Let $\theta_{t,x}$ denote the Young family of a weak limit point Θ of the sequence of Young measures Θ_δ as $\delta \to 0$, then for couples (h_1, J_1) and (h_2, J_2) of Lax entropy pairs we have a compound factorization property:

$$\theta_{t,x}(h_1 J_2) - \theta_{t,x}(h_2 J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) - \theta_{t,x}(h_2)\theta_{t,x}(J_1) \tag{3.1}$$

almost everywhere on $\mathbb{R}^2$. In his pioneering papers Ronald DiPerna managed to show that (3.1) implies the Dirac property of the Young family, at least if the sequence of approximate solutions is uniformly bounded, see DiPerna (1985) with further references.

The microscopic evolution: The Ito lemma yields a parabolic energy inequality

$$\begin{aligned}\partial_t \mathsf{E} H_k(\omega(t)) = {} & \mathsf{E}\left(p_{k+1}V'(r_k) - p_k V'(r_{k-1})\right) \\ & + \sigma(\varepsilon)\mathsf{E}\left(p_k\left(p_{k+1} + p_{k-1} - 2p_k\right)\right) \\ & + \sigma(\varepsilon)\mathsf{E}\left(V'(r_k)(V'(r_{k+1}) + V'(r_{k-1}) - 2V'(r_k))\right)\end{aligned}$$

at the microscopic level. If $\varepsilon\sigma(\varepsilon)$ remains positive as $\varepsilon \to 0$, then the tightness in the local topology of $L^2(\mathbb{R})$ of the distributions of the time averaged process might follow from this bound in much the same way as it is done in PDE theory.[5] However, $\varepsilon\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, thus the bound degenerates in the limit, consequently there is no hope to get tightness in L^2. That is why we say that a direct compactness argument does not work, the method of compensated compactness is needed.

In our case a difficult step of the usual non-gradient analysis can be avoided by considering the Lax entropy pairs (h, J) as functions of the block averaged empirical process $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$. Entropy production $X_\varepsilon := \partial_t h(\hat{u}_\varepsilon, \hat{v}_\varepsilon) + \partial_x J(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ is defined as a generalized function, without the condition $\varepsilon\sigma^2(\varepsilon) \to +\infty$ its fluctuations might explode in the limit even if we define the empirical processes in terms of block averages. The main difficulty is to identify the macroscopic flux in the microscopic expression of $\mathcal{L}_0 h$, and to show that the remainders do vanish in the limit. This is achieved by replacing block averages of the microscopic currents of momenta with their equilibrium expectations, a logarithmic Sobolev inequality plays a decisive role in the computations. This substitution transforms the evolution equation of h into a fairly transparent form: we can recover essentially the same structure which appears when the vanishing viscosity limit for (2.8) is performed. At this point can we launch the stochastic theory of compensated compactness, and the proof is terminated by referring to known results from PDE theory. Unfortunately we can not find bounded, *positively invariant regions* in stochastic situations as DiPerna (1985) did at the PDE level, but the results of Shearer (1994) and Serre–Shearer (1994) on an L^p theory of compensated compactness are applicable.

[5]In case of the diffusive models of Fritz (1986) and its continuations, an energy inequality implies this kind of tightness of the process in the space of trajectories. Guo–Papanicolau–Varadhan (1988) had raised the problem to the level of measures μ_t, and instead of energy and the H^{+1} norm of configurations, the relative entropy and its rate of production (Dirichlet form) are estimated to get the required a priori bounds including an energy inequality.

3.3. Stochastic theory of compensated compactness

Most computations involve *mesoscopic block averages* of size $l = l(\varepsilon)$ such that

$$\lim_{\varepsilon \to 0} \frac{l(\varepsilon)}{\sigma(\varepsilon)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\varepsilon l^3(\varepsilon)}{\sigma(\varepsilon)} = +\infty.$$

For sequences ξ_k indexed by $\mathbb{Z}$ we define two kinds of block averages:

$$\bar{\xi}_{l,k} := \frac{1}{l} \sum_{j=0}^{l-1} \xi_{k-j} \quad \text{and} \quad \hat{\xi}_{l,k} := \frac{1}{l^2} \sum_{j=-l}^{l} (l - |j|)\, \xi_{k+j}. \tag{3.2}$$

For example, $\bar{V}'_{l,k}$ denotes the arithmetic mean of the sequence $\xi_j = V'(r_j)$. We start calculations with the "smooth" averages $\hat{\xi}_{l,k}$, the arithmetic means appear in canonical expectations. The corresponding empirical process $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$ and $(\bar{u}_\varepsilon\, \bar{v}_\varepsilon)$ are defined according to $\hat{u}_\varepsilon(t,x) := \hat{p}_{l,k}(t/\varepsilon)$ if $|\varepsilon k - x| < \varepsilon/2$, and so on. Since $\hat{u}_\varepsilon$ and $\hat{v}_\varepsilon$ are bounded in a mean sense in $L^2(dt, dx)$, the distributions $\hat{\mathsf{P}}_\varepsilon$ of the Young measure Θ form a tight family; these are now defined as $d\Theta_\varepsilon := dt\, dx\, \theta^\varepsilon_{t,x}(du)$, where $\theta^\varepsilon_{t,x}$ is the Dirac mass at the actual value of $(\hat{u}_\varepsilon, \hat{v}_\varepsilon)$. The Young family controls the asymptotic behavior of various functions of the empirical processes.

Given a Lax entropy pair (h, J), the associated entropy production is defined as

$$X_\varepsilon(\varphi, h) := -\int_0^\infty \int_{-\infty}^\infty \left(h(\hat{u}_\varepsilon, \hat{v}_\varepsilon)\varphi'_t(t,x) + J(\hat{u}_\varepsilon, \hat{v}_\varepsilon)\varphi'_x(t,x)\right) dx\, dt,$$

where the test function φ is compactly supported in the interior of $\mathbb{R}^2_+$. We call an entropy pair (h, J) *well controlled* if its entropy production decomposes as $X_\varepsilon(\varphi, h) = Y_\varepsilon(\varphi, h) + Z_\varepsilon(\varphi, h)$, and we have two random functionals $A_\varepsilon(\phi, h)$ and $B_\varepsilon(\phi, h)$ such that

$$|Y_\varepsilon(\psi\phi, h)| \le A_\varepsilon(\phi, h)\|\psi\|_+ \quad \text{and} \quad |Z_\varepsilon(\psi, h)| \le B_\varepsilon(\phi, h)\|\psi\|,$$

where $\|\cdot\|$ is the uniform norm, while $\|\cdot\|_+$ denotes the norm of the Sobolev space H^{+1}. Here the test function ϕ is compactly supported in the interior of $\mathbb{R}^2_+$, its role is to localize the problem. The factors A_ε and B_ε do not depend on ψ, moreover $\lim \mathsf{E} A_\varepsilon(\phi, h) = 0$ and $\limsup \mathsf{E} B_\varepsilon(\phi, h) < +\infty$ as $\varepsilon \to 0$.

Proposition 3.2. *If (h_1, J_1) and (h_2, J_2) are well controlled entropy pairs, then* (3.1) *holds true with probability one with respect to any limit distribution of $\hat{\mathsf{P}}_\varepsilon$ that we obtain as $\varepsilon \to 0$.*

This is the stochastic version of the Div-Curl Lemma above. The proof is not difficult, by means of the *Skorohod Embedding Theorem* it can be reduced to the original, deterministic version, see Fritz (2001), Fritz (2004) and Fritz–Tóth (2004). The main problem is the verification of its conditions, the logarithmic Sobolev inequality plays an essential role here.

3.4. The a priori bounds

Following Fritz (1990), our a priori bounds are all based on the next inequality that controls relative entropy and its rate of production. The initial condition implies that

$$\mathsf{S}[\mu_{t,\varepsilon,n}|\lambda_{0,0}] + \sigma(\varepsilon)\int_0^t D[\mu_{s,\varepsilon,n}|\lambda_{0,0}]\,ds \le C\left(t + \sqrt{n^2+\sigma(\varepsilon)t}\right)$$

for all t,n,ε with the same constant C, where D is the Dirichlet form, it is due to the elliptic perturbation of the anharmonic chain:

$$D[\mu_{t,\varepsilon,n}|\lambda_{0,0}] := \sum_{k=-n}^{n-1}\int(\nabla_1\partial_k\sqrt{f_n})^2\,d\lambda + \sum_{k=-n}^{n-1}\int(\nabla_1\tilde{\partial}_k\sqrt{f_n})^2\,d\lambda,$$

where $\nabla_l\xi_k := (1/l)(\xi_{k+l}-\xi_k)$, $f_n := d\mu_{t,\varepsilon,n}/d\lambda_{0,0}$, $\partial_k := \partial/\partial p_k$ and $\tilde{\partial}_k := \partial/\partial r_k$. This is the consequence of a system of differential inequalities:

$$\partial_t S_n + 2\sigma(\varepsilon)D_n \le K\left(S_{n+1}-S_n+\sigma(\varepsilon)\sqrt{S_{n+1}-S_n}\,\sqrt{D_{n+1}-D_n}\right),$$

where $S_n := \mathsf{S}[\mu_{t,\varepsilon,n}|\lambda_{0,0}]$ and $D_n := D[\mu_{t,\varepsilon,n}|\lambda_{0,0}]$ for brevity. For a proof of this local entropy bound see Fritz (2011) with further references.

LSI: The logarithmic Sobolev inequality we are going to use, can be stated as follows. Given $\bar{r}_{l,k} = v$, let $\mu^v_{l,k}$ and $\lambda^v_{l,k}$ denote the conditional distributions of the variables $r_k, r_{k+1}, ..., r_{k+l-1}$ with respect to μ and $\lambda_{0,0}$, and set $f^v_{l,k} := d\mu^v_{l,k}/d\lambda^v_{l,k}$, then

$$\int \log f^v_{l,k}\,d\mu^v_{l,k} \le l^2 C_{\mathrm{lsi}}\sum_{j=k}^{k+l-2}\int\left(\nabla_1\tilde{\partial}_k(f^v_{l,k})^{1/2}\right)^2\,d\lambda^v_{l,k}$$

for all μ, v, k, l with a universal constant C_{lsi} depending only on V. Of course, a similar inequality holds true for the conditional distributions of momenta. Combining this with the standard entropy inequality $\int\varphi\,d\mu \le \mathsf{S}[\mu|\lambda] + \log\int e^\varphi\,d\lambda$, the calculation of expectations reduces to large deviation bounds for the canonical distributions of the equilibrium measure $\lambda_{0,0}$. The most important consequence of the local entropy bound and this LSI is the evaluation of the microscopic current of momentum as follows:

$$\sum_{|k|<n}\int_0^t\int\left(\bar{V}'_{l,k} - S'(\bar{r}_{l,k})\right)^2\,d\mu_{s,\varepsilon}\,ds \le C_1\left(\frac{nt}{l} + \frac{l^2\sqrt{n^2+\sigma(\varepsilon)t}}{\sigma(\varepsilon)}\right).$$

Similar bounds control the differences $\bar{r}_{l,k+l} - \bar{r}_{l,k}$ and $\hat{r}_{l,k} - \bar{r}_{l,k}$. Later on the validity of such a bound will be indicated as $\bar{V}'_{l,k} \approx S'(\bar{r}_{l,k})$, $\bar{r}_{l,k+l} \approx \bar{r}_{l,k}$, and so on.

Entropy flux: Finally, let us outline the crucial step of the evaluation of entropy production at a heuristic level. Consider a Lax entropy $h = h(u,v)$ with flux

$J = J(u, v)$ and expand J. The second order terms of the Lagrange expansion can be neglected, thus we have

$$\begin{aligned}X_{0,k} &:= \mathcal{L}_0 h(\hat{p}_{l,k}, \hat{r}_{l,k}) + J(\hat{p}_{l,k+1}, \hat{r}_{l,k+1}) - J(\hat{p}_{l,k}, \hat{r}_{l,k}) \\ &\approx h'_u(\hat{p}_{l,k}, \hat{r}_{l,k})(\hat{V}'_{l,k} - \hat{V}'_{l,k-1}) + h'_v(\hat{p}_{l,k}, \hat{r}_{l,k})(\hat{p}_{l,k+1} - \hat{p}_{l,k}) \\ &\quad + J'_u(\hat{p}_{l,k}, \hat{r}_{l,k})(\hat{p}_{l,k+1} - \hat{p}_{l,k}) + J'_v(\hat{p}_{l,k}, \hat{r}_{l,k})(\hat{r}_{l,k+1} - \hat{r}_{l,k}).\end{aligned}$$

Since $h'_u(u, v)S''(v) + J'_v(u, v) = h'_v(u, v) + J'_u(u, v) = 0$,

$$X_{0,k} \approx h'_u(\hat{p}_{l,k}, \hat{r}_{l,k})(\hat{V}'_{l,k} - \hat{V}'_{l,k-1}) - h'_u(\hat{p}_{l,k}, \hat{r}_{l,k})S''(\hat{r}_{l,k})(\hat{r}_{l,k+1} - \hat{r}_{l,k}).$$

Observe now that $\hat{\xi}_{l,k+1} - \hat{\xi}_{l,k} = (1/l)(\bar{\xi}_{l,k+l} - \bar{\xi}_{l,k})$, thus the substitution $\bar{V}'_{l,k} \approx S'(\bar{r}_{l,k})$ results in $l\, X_{0,k} \approx 0$ as

$$l\, X_{0,k} \approx h'_u(\hat{p}_{l,k}, \hat{r}_{l,k}) \left(S'(\bar{r}_{l,k-1+l}) - S'(\bar{r}_{l,k-1}) - S''(\hat{r}_{l,k})(\bar{r}_{l,k+l} - \bar{r}_{l,k})\right).$$

Of course, the precise computation is much more complicated because in the formula X_ε of entropy production the terms $X_{0,k}$ have a factor $1/\varepsilon$. In fact, $(\varepsilon l(\varepsilon)\sigma(\varepsilon))^{-1}$ is the order of the replacement error; that is why we need $\varepsilon\sigma^2(\varepsilon) \to +\infty$ and the sharp explicit bounds provided by the logarithmic Sobolev inequality.

4. Relaxation of interacting exclusions

We consider ± 1 charges in an electric field, positive charges jump to the right on $\mathbb{Z}$, negative charges move to the left with unit jump rates in both cases such that two or more particles can not coexist at the same site. There is an interaction between these processes: if charges of opposite sign meet, then they jump over each other at rate 2. The configurations are doubly infinite sequences $\omega_k \in \{-1, 0, 1\}$ indexed by $\mathbb{Z}$, $\omega_k = 0$ indicates an empty site, and $\eta_k := \omega_k^2$ denotes the occupation number. The generator of the process is acting on local functions φ as

$$\mathcal{L}_0 \varphi(\omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (\eta_k + \eta_{k+1} + \omega_k - \omega_{k+1})(\varphi(\omega^{k,k+1}) - \varphi(\omega));$$

$\omega \to \omega^{k,k+1}$ indicates the exchange of ω_k and ω_{k+1}. This most interesting model had been introduced by Tóth–Valkó (2003), where its HDL in a smooth regime is demonstrated, too. The total charge $P = \sum \omega_k$ and particle number $R = \sum \eta_k$ are obviously preserved by the evolution, and the associated family of translation invariant stationary product measures $\{\lambda_{u,\rho}\}$ can be parametrized so that $\int \omega_k \, d\lambda_{u,\rho} = u$ and $\int \eta_k \, d\lambda_{u,\rho} = \rho$. Conservation of ω and η means that they are driven by currents, i.e. $\mathcal{L}_0 \omega_k = \mathrm{j}^\omega_{k-1} - \mathrm{j}^\omega_k$ and $\mathcal{L}_0 \eta = \mathrm{j}^\eta_{k-1} - \mathrm{j}^\eta_k$, where

$$\begin{aligned}\mathrm{j}^\omega_k &:= (1/2)\left(\eta_k + \eta_{k+1} - 2\omega_k\omega_{k+1} + \omega_k\eta_{k+1} - \eta_k\omega_{k+1} + \eta_k - \eta_{k+1}\right), \\ \mathrm{j}^\eta_k &:= (1/2)\left(\omega_k + \omega_{k+1} - \omega_k\eta_{k+1} - \eta_k\omega_{k+1} + \eta_k - \eta_{k+1}\right).\end{aligned}$$

Since $\int \mathrm{j}_k^\omega \, d\lambda_{u,\rho} = \rho - u^2$ and $\int \mathrm{j}_k^\eta \, d\lambda_{u,\rho} = u - u\rho$, the principle of local equilibrium suggests that under hyperbolic scaling a version of the Leroux system:

$$\partial_t u + \partial_x(\rho - u^2) = 0 \quad \text{and} \quad \partial_t \rho + \partial_x(u - u\rho) = 0 \tag{4.1}$$

governs the macroscopic evolution. The strong ergodic hypothesis can easily be proven by a standard entropy argument. In a regime of shock waves the method of compensated compactness is applied to derive the Leroux system; therefore an additional stirring mechanism:

$$\mathcal{G}_e \varphi(\omega) := \sum_{k \in \mathbb{Z}} \left(\varphi(\omega^{k,k+1}) - \varphi(\omega) \right)$$

is needed to regularize the process. The full generator reads as $\mathcal{L} := \mathcal{L}_0 + \sigma(\varepsilon)\, \mathcal{G}_e$, and our usual conditions $\varepsilon\sigma(\varepsilon) \to 0$ and $\varepsilon\sigma^2(\varepsilon) \to +\infty$ are assumed.

The statement is similar to the case of isentropic elastodynamics, the proof is based on the logarithmic Sobolev inequality what we have for the stirring generator $\mathcal{G}_e$, see Fritz–Tóth (2004), where HDL is proven in a periodic setting. The extension of this result to general initial values is explained by Fritz–Nagy (2006), the optimal version concerns the mollified empirical processes $\hat{u}_\varepsilon(t,x) := \hat{\omega}_{l,k}(t/\varepsilon)$ and $\hat{\rho}_\varepsilon(t,x) := \hat{\eta}_{l,k}(t/\varepsilon)$ if $|\varepsilon k - x| < \varepsilon/2$, where the block size $l = l(\varepsilon)$ is the same as in Section 3.

Theorem 4.1. *The distributions of our empirical processes form a tight family with respect to the strong local topology of* $L^1(\mathbb{R}_+^2)$*, and any limit distribution of* $(\hat{u}_\varepsilon, \hat{\rho}_\varepsilon)$ *is concentrated on a set of weak solutions to* (4.1). *These weak solutions satisfy the Lax entropy condition, too.*

A uniqueness theorem for the Leroux system requires only a local bound on the total variation of the weak solution we have constructed, nevertheless we are not able to prove the uniqueness of the hydrodynamic limit.

4.1. Creation and annihilation of charges

In the paper Fritz–Nagy (2006) it was shown that an additional spin-flip mechanism relaxes the Leroux system to the Burgers equation $\partial_t \rho + \kappa\, \partial_x(\rho - \rho^2) = 0$ even in the case of shocks, where $\kappa = 0$ in the *completely symmetric* case. The replacement $u \approx \kappa\rho$ is due to a second logarithmic Sobolev inequality. The following modification of the above process of interacting exclusions is a caricature of *electrophoresis,* and it is interesting also from the point of view of mathematics because the PDE method of *relaxation schemes* is reformulated for the microscopic dynamics.

The model: Imagine that when two particles of opposite charge collide, then instead of jumping over each other, they may kill each other and disappear, while at two neighboring empty sites a couple $(+1, -1)$ can be created. The action $\omega \to \omega^{k+}$ of *creation* at the bond $b = (k, k+1)$ means that $(\omega_k, \omega_{k+1}) \to (1, -1)$ if $\omega_k = \omega_{k+1} = 0$, while *annihilation* $\omega \to \omega^{k-}$ is defined by $(\omega_k, \omega_{k+1}) \to (0, 0)$

if $\omega_k = 1$ and $\omega_{k+1} = -1$; at other sites the configuration is not altered. The generator of this process of *interacting exclusions with creation and annihilation* reads as $\mathcal{L}^* = \mathcal{L}_0 + \beta(\varepsilon)\,\mathcal{G}^*$, where

$$\mathcal{G}^*\varphi(\omega) := \sum_{k\in\mathbb{Z}}(1-\eta_k)(1-\eta_{k+1})(\varphi(\omega^{k+})-\varphi(\omega)) + \frac{1}{4}\sum_{k\in\mathbb{Z}}(\eta_k+\omega_k)(\eta_{k+1}-\omega_{k+1})(\varphi(\omega^{k-})-\varphi(\omega)).$$

Since we do not want to postulate smoothness of the macroscopic solution, the process should be regularized by stirring, thus the effective generator becomes $\mathcal{L} := \mathcal{L}^* + \sigma(\varepsilon)\,\mathcal{G}_e$. The factor $\sigma = \sigma(\varepsilon)$ is the same as above, and it is natural to assume that β is a positive constant because it is the parameter of the basic model.

Creation-annihilation violates the conservation of particle number, only total charge $\sum \omega_k$ is preserved by our stochastic dynamics. A product measure $\lambda_{u,\rho}$ will be stationary if $\lambda_{u,\rho}[\omega_k = 0, \omega_{k+1} = 0] = \lambda_{u,\rho}[\omega_k = 1, \omega_{k+1} = -1]$, that is $4(1-\rho)^2 = (\rho^2 - u^2)$, whence

$$\rho = F(u) := (1/3)(4 - \sqrt{4-3u^2}) \tag{4.2}$$

is the criterion of stationarity because the second root:

$$\tilde{F}(u) := (1/3)(4 + \sqrt{4-3u^2}) \geq 5/3 > 1.$$

Therefore our one-parameter family $\{\lambda_u^* : |u| < 1\}$ of stationary product measures is defined by $\lambda_u^* := \lambda_{u,F(u)}$. Of course, $\int \omega_k\, d\lambda_u^* = u$ and $\int \eta_k\, d\lambda_u^* = F(u)$, thus $\int \mathrm{j}_k^\omega\, d\lambda_u^* = F(u) - u^2$. On the other hand, $\mathcal{G}^*\omega_k = \mathrm{j}_{k-1}^{\omega*} - \mathrm{j}_k^{\omega*}$ is a difference of currents,

$$\mathrm{j}_k^{\omega*}(\omega) := (1/4)(\eta_k+\omega_k)(\eta_{k+1}-\omega_{k+1}) - (1-\eta_k)(1-\eta_{k+1}), \tag{4.3}$$

and $\int \mathrm{j}_k^{\omega*}\, d\lambda_{u,\rho} = C(u,\rho) := (3/4)(\rho - F(u))(\tilde{F}(u) - \rho)$, thus the equilibrium expectation of $\mathrm{j}_k^{\omega*}$ does vanish, consequently the principle of local equilibrium predicts

$$\partial_t u(t,x) + \partial_x(F(u) - u^2) = 0 \tag{4.4}$$

as the result of the hyperbolic scaling limit. Note that the flux is neither convex nor concave, thus the structure of shock waves may be rather complex.

It is not a surprise that the contribution of the creation-annihilation mechanism does not appear in the limit. The generator $\mathcal{G}^*$ is symmetric in $L^2(d\lambda_u^*)$, consequently a diffusive scaling would be needed to recover its action.

Main result. Assume that the initial distributions satisfy

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{k\in\mathbb{Z}} \varphi(\varepsilon k)\omega_k(0) = \int_{-\infty}^{\infty} \psi(x) u_0(x)\, dx$$

in probability for all compactly supported $\varphi \in C(\mathbb{R})$. We say that a measurable and bounded $u = u(t,x)$ is a *weak entropy solution* to (4.4) with initial value u_0 if

$$\int_0^\infty \int_{-\infty}^\infty (\psi_t'(t,x)u(t,x) + \psi_x'(t,x)(F(u(t,x)) - u^2(t,x)))\,dx\,dt$$
$$+ \int_{-\infty}^\infty \psi(0,x)u_0(x)\,dx = 0,$$

and for all convex entropy pairs (h, J) we have the Lax inequality:

$$-X_\varepsilon(\psi,h) = \int_0^\infty \int_{-\infty}^\infty (\psi_t'(t,x)h(u) + \psi_x'(t,x)J(u))\;dx\,dt \;\geq 0 \tag{4.5}$$

whenever $0 \leq \psi \in C^1(\mathbb{R}^2)$ is compactly supported in the interior of $\mathbb{R}^2_+$. Entropy pairs of (4.4) are characterized by $J'(u) = (F'(u) - 2u)h'(u)$, that is $\partial_t h(u) + \partial_x J(u) = 0$ along classical solutions. Our effective empirical process $\hat{u}_\varepsilon$ is now defined as $\hat{u}_\varepsilon(t,x) := \hat{\omega}_{l,k}(t/\varepsilon)$ if $|\varepsilon k - x| < \varepsilon/2$; the mesoscopic block size $l = l(\varepsilon)$ is just as big as it was in the previous section.

In the paper by Bahadoran–Fritz–Nagy (2011) we prove:

Theorem 4.2. *The above conditions imply that*

$$\lim_{\varepsilon\to 0} \mathsf{E} \int_0^\tau \int_{-r}^r |u(t,x) - \hat{u}_\varepsilon(t,x)|\,dx\,dt = 0$$

for all $r, \tau > 0$, where u is the uniquely specified weak entropy solution to (4.4) *with initial value u_0.*

Let us remark that the coefficient $\beta > 0$ needs not be a constant, it is sufficient to assume that $\sigma(\varepsilon)\beta(\varepsilon) \to +\infty$ and $\varepsilon\sigma^2(\varepsilon)\beta^{-4}(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

4.2. Relaxation in action

The proof follows the standard technology of the stochastic theory of compensated compactness, the entropy production for entropy pairs of (4.4) has to be evaluated. Here the uniqueness of the hydrodynamic limit is a consequence of the Lax entropy inequality, see Chen–Rascle (2000), thus $\limsup X_\varepsilon(\psi,h) \leq 0$ is also needed for $\psi \geq 0$ and convex h. We are facing with the computation of four basic quantities, besides j_k^ω, j_k^η and $\mathrm{j}_k^{\omega *}$,

$$\begin{aligned}\mathcal{G}^*\eta_k &= (1-\eta_k)(1-\eta_{k+1}) - (1/4)(\eta_k + \omega_k)(\eta_{k+1} - \omega_{k+1})\\ &\quad + (1-\eta_{k-1})(1-\eta_k) - (1/4)(\eta_{k-1} + \omega_{k-1})(\eta_{k-1} - \omega_{k-1})\end{aligned}$$

should also be evaluated. Since $\mathcal{G}^*\eta_k = -\mathfrak{j}^{\omega*}_{k-1} - \mathfrak{j}^{\omega*}_k$, we have

$$\int \mathcal{G}^*\eta_k \, d\lambda_{u,\rho} = (3/2)(\rho - F(u))(\rho - \tilde{F}(u)) = -2C(u,\rho). \tag{4.6}$$

The macroscopic flux: The fundamental local bound on relative entropy and its rate of production holds true also in this case, see Lemma 3.1 of our paper, thus the logarithmic Sobolev inequality involving the Dirichlet form of $\mathcal{G}_e$ applies, too. In this way we can estimate canonical expectations given $\bar{\omega}_{l,k}$ and $\bar{\eta}_{l,k}$, see Lemmas 3.3–3.5 in Bahadoran–Fritz–Nagy (2011); the explicit upper bounds are the same as in Section 3.4. Therefore the replacements

$$\bar{\mathfrak{j}}^{\omega}_{l,k} \approx \bar{\eta}_{l,k} - (\bar{\omega}_{l,k})^2, \quad \bar{\mathfrak{j}}^{\eta}_{l,k} \approx \bar{\omega}_{l,k} - \bar{\omega}_{l,k}\bar{\eta}_{l,k}, \quad \bar{\mathfrak{j}}^{\omega*}_{l,k} \approx C(\bar{\omega}_{k,l}, \bar{\eta}_{l,k}) \tag{4.7}$$

and $\bar{\eta}^*_{l,k} \approx -2C(\bar{\omega}_{k,l}, \bar{\eta}_{l,k})$, where $\eta^*_j := \mathcal{G}^*\eta_j$ for convenience, are all allowed, moreover $\bar{\omega}_{l,k+l} \approx \bar{\omega}_{l,k} \approx \hat{\omega}_{l,k}$ and $\bar{\eta}_{l,k+l} \approx \bar{\eta}_{l,k}$.

Entropy production: Since $\mathcal{G}^*$ is reversible, the critical component of entropy production is induced by $\mathcal{L}_0$. Let us consider now an entropy pair (h, J) of (4.4), i.e. $J'(u) = (F'(u) - 2u)h'(u)$. In view of the asymptotic equivalence relations listed above, we obtain that

$$\begin{aligned}
X^*_{0,k} &:= \mathcal{L}_0 h(\hat{\omega}_{l,k}) + J(\hat{\omega}_{l,k+1}) - J(\hat{\omega}_{l,k}) \approx h'(\hat{\omega}_{l,k})(\hat{\mathfrak{j}}^{\omega}_{l,k-1} - \hat{\mathfrak{j}}^{\omega}_{l,k}) \\
&\quad + (F'(\hat{\omega}_{l,k}) - 2\hat{\omega}_{l,k})\, h'(\hat{\omega}_{l,k})(\hat{\omega}_{l,k+1} - \hat{\omega}_{l,k}) \\
&\approx (1/l)h'(\hat{\omega}_{l,k})\left(\bar{\eta}_{l,k} - \bar{\eta}_{l,k+l} - (\bar{\omega}_{l,k})^2 + (\bar{\omega}_{l,k+l})^2\right) \\
&\quad + (1/l)h'(\hat{\omega}_{l,k})\left(F'(\bar{\omega}_{l,k}) - \bar{\omega}_{l,k} - \bar{\omega}_{l,k+l}\right)(\bar{\omega}_{l,k+l} - \bar{\omega}_{l,k}) \\
&\approx (1/l)h'(\hat{\omega}_{l,k})\left(\bar{\eta}_{l,k} - \bar{\eta}_{l,k+l} + F'(\bar{\omega}_{l,k})(\bar{\omega}_{l,k+l} - \bar{\omega}_{l,k})\right),
\end{aligned}$$

whence the required $l\,X^*_{0,k} \approx 0$ would follow by the substitution $\bar{\eta}_{l,k} \approx F(\bar{\omega}_{l,k})$. Since we do not have the appropriate logarithmic Sobolev inequality, another tool must be found.

Relaxation schemes: Observe that $\bar{\eta}_{l,k}$ appears with a negative sign in the formula of $\mathcal{G}^*\bar{\eta}_{l,k}$, see also (4.6), thus there is a hope to experience *relaxation*, which results in $C(\bar{\omega}_{l,k}, \bar{\eta}_{l,k}) \to 0$ as $\varepsilon \to 0$. Although the evolution equations of $\bar{\omega}_{l,k}$ and $\bar{\eta}_{l,k}$ are rather complicated, the following couple of approximate identities reflects quite well the underlying structure. Applying the substitution relations (4.7) and neglecting obviously vanishing terms, we get

$$\begin{aligned}
&d\tilde{u}_\varepsilon + \partial_x(\tilde{\rho}_\varepsilon - \tilde{u}_\varepsilon^2)\,dt + \beta\,\partial_x C(\tilde{u}_\varepsilon, \tilde{\rho}_\varepsilon)\,dt \approx 0, \\
&d\tilde{\rho}_\varepsilon + \partial_x(\tilde{u}_\varepsilon - \tilde{u}_\varepsilon\tilde{\rho}_\varepsilon) + (2\beta/\varepsilon)\,C(\tilde{u}_\varepsilon, \tilde{\rho}_\varepsilon)\,dt \approx 0,
\end{aligned}$$

where $\tilde{u}_\varepsilon \sim \bar{\omega}_{l,k}$ and $\tilde{\rho}_\varepsilon \sim \bar{\eta}_{l,k}$ by mollification. Since

$$(\rho - F(u))C(u,\rho) \geq \Psi(u,\rho) := (1/2)\left(\rho - F(u)\right)^2,$$

even the trivial Liapunov function Ψ can be applied to conclude that $\bar{\eta}_{l,k} \approx F(\bar{\omega}_{l,k})$. This trick works well if $\varepsilon\sigma^2(\varepsilon)\beta^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, a slightly better result can be proven by replacing Ψ with a clever Lax entropy, see Bahadoran–Fritz–Nagy (2011).

5. Concluding remarks

In spite of some progress in the stochastic theory of compensated compactness, there are many relevant open problems whose solution seems to be hard or even hopeless at this time.

The Lax inequality: The dominant term of entropy production $X_\varepsilon(\psi, h)$ is generated by the elliptic components of $\mathcal{L} = \mathcal{L}_0 + \sigma(\varepsilon)\mathcal{G}_p + \sigma(\varepsilon)\mathcal{G}_r$. It is bounded in the space of measures, and the contribution of $\mathcal{G}_p$ is obviously not positive if $\psi \geq 0$ and h is convex. Our naive large deviation technique is not strong enough to exploit that V is convex. The Lax inequality restricts the set of limiting weak solutions, but in the case of systems it is not a known condition of uniqueness.

Uniqueness of HDL: This is a very hard problem in the case of a couple of conservation laws because any criterion of uniqueness presupposes a sharp local bound at fixed times. Unfortunately, in the case of stochastic models we are able to bound expectations of space-time integrals only. From the point of view of computations the microscopic systems of statistical physics are more complicated than the sophisticated numerical schemes of PDE theory.[6] For example, even the existence of positively invariant regions is a problematic issue.

Physical viscosity: It would be nice to materialize the argumentation of Serre–Shearer (1994) at the microscopic level, that is to consider hyperbolic scaling of the model $\mathcal{L} = \mathcal{L}_0 + \sigma\mathcal{G}_p$ in a regime of shocks. This is not easy because the Dirichlet form of $\mathcal{G}_p$ controls the distribution of velocities only, while the most crucial step consists of the substitution $\bar{V}'_{l,k} \approx S'(\bar{r}_{l,k})$. The less interesting case of $\mathcal{L} = \mathcal{L}_0 + \sigma\mathcal{G}_r$ seems to be simpler, but it not trivial at all.

The strength of artificial viscosity: The condition $\varepsilon\sigma^2(\varepsilon) \to +\infty$ is not necessary in the case of attractive models, but it is systematically applied in more general situations.

Euler equations with physical viscosity: HDL of the model $\mathcal{L} = \mathcal{L}_0 + (1/\varepsilon)\mathcal{G}_r$ results in the p-system of elastodynamics with artificial viscosity, see Theorem 3 in Fritz (1990). The derivation of the viscid Euler equations (1.5) of Chen–Dafermos (1995) is more complicated because then a momentum and energy preserving diffusive noise should be added to the equations of the anharmonic chain. To solve the resulting non-gradient problem, the spectral gap of the elliptic components of the generator ought to be determined.

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